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The method of characteristics in the theory of resonant or nonresonant nonlinear optics

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Abstract. The method of characteristics is used to analyse the system of five nonlinear partial differential equations describing ultrashort optical pulse propagation through a medium of two-level atoms. Contrary to previous suggestions it is shown rigorously that the system is causal: no physical speeds exceed c. The method is well adapted to numerical integration. It can take full account of both back scattering and the effect of a surface to the dielectric. A number of new physical results obtained this way are briefly reported.

1. Introduction

It has been noted (Basov *et al* 1966, Icsevgi and Lamb 1969, Bullough 1970 to be referred to as I, Bullough and Ahmad 1971 to be referred to as II) that there are analytical solutions of the equations assumed for ultrashort ultra-intense optical pulse propagation in an amplifying medium of two-level atoms in which the pulse (ie group) velocity exceeds the velocity of light *in vacuo* (c). This is contrary to physical understanding and Icsevgi and Lamb (1969) in particular adduce physical arguments and some numerical examples which help to exclude the possibility. The main purpose of this note is to provide a rigorous proof that physically acceptable pulses cannot have such propagation speeds. Precisely we prove (i) causality: the current state is uniquely determined by initial data prescribed at a previous time, (ii) that relevant initial data lie within the light cone and (iii) that every wave front penetrating from a vacuum into a medium of two-level atoms, resonant or not, travels *inside the medium* at the velocity of light *in vacuo* (c).

The method we use is Riemann's method of characteristics. The theory is set out in § 2 and the proof of causality follows in § 3. This method has scarcely been used before in theoretical nonlinear optics; da Costa (1970) uses characteristic theory in heuristic arguments for causality, and DeMartini *et al* (1967) have used characteristics in a theory of pulse steepening. Here we use the theory to give a fundamental analysis of optical pulse propagation; in § 4 we comment by example on the theory of optical shocks and in § 5 we consider the proper posing of initial conditions. In § 6 we indicate briefly how the method may be used for direct numerical integration of the exact equations for a smooth dielectric of two-level atoms with a boundary.

2. Characteristic theory of the nonlinear optics equations

Because characteristics theory has received little application in quantum optics, we shall first define what we mean by the characteristics of a system of partial differential equations

(PDE): we use the analysis of Courant and Lax (Courant and Lax 1949, Forsythe and Wasow 1960, Ames 1969). In general any set of higher order partial differential equations in two independent variables z, t can be rewritten, with the aid of supplementary unknown functions, as a set of first order PDE in the 'quasilinear' form ($u_z \equiv \partial u/\partial z, u_t \equiv \partial u/\partial t$)

$$\sum_{i=1}^{N} (a^{\nu i}u_z^i + b^{\nu i}u_t^i) + d^\nu = 0 \qquad \nu = 1, 2, \ldots, N. \tag{1}$$

The u^i are the functions of (z, t) to be determined. The $N \times N$ square matrices $a^{\nu i}$ and $b^{\nu i}$ and the vector d^ν are in general functions of (z, t) *and* the u^i. In the special case where $a^{\nu i}$ and $b^{\nu i}$ are independent of the u^i so that the nonlinearity lies in the d^ν, the equations (1) are said to be semilinear.

In quasilinear form each equation contains a mixture of differentials of different functions taken in different directions in the z, t plane. By multiplying equation (1) by a suitable matrix $t^{\mu\nu}$ we obtain the equations in a form in which each function u^i in any one equation appears differentiated in the same direction

$$\sum_{i}^{N} a^{*\nu i}(c^\nu u_z^i + u_t^i) + d^{*\nu} = 0 \qquad \nu = 1, 2, \ldots, N. \tag{2}$$

For each equation, the differential of each u^i lies along the line with equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = c^\nu \qquad \nu = 1, 2, \ldots, N. \tag{3}$$

The solution of this system of ordinary DE, including N arbitrary constants, defines an N fold family of curves: these are the characteristic curves of the system of PDE (1). Generally the c^ν are functions of (z, t) *and* the u^i: the characteristics depend on the particular solution, and are therefore 'movable' in this sense. But if the PDE (1) are semilinear the c^ν depend on (z, t) only and the characteristics from equation (3) are 'fixed', that is the same for all solutions. This is fortunately the case with the system studied here.

The two-level atom has a pseudospin representation. The simplest equations describing the propagation of plane EM waves in a dielectric of two-level atoms are (I and II, McCall and Hahn 1969) the Bloch type equation

$$\boldsymbol{r}_t(z, t) = \boldsymbol{\omega}(z, t) \times \boldsymbol{r}(z, t) \tag{4}$$

for the atoms themselves and the Maxwell wave equation

$$E_{zz}(z, t) - c^{-2}E_{tt}(z, t) = 4\pi c^{-2} n\, ex_{0s}(r_1(z, t))_{tt} \tag{5}$$

which couples the atoms via their radiated fields. The components of $\boldsymbol{r}$ relate to the elements of the density matrix for a single effective atom at z by

$$r_1(z, t) \equiv \rho_{s0}(z, t) + \rho_{0s}(z, t) \qquad r_2 \equiv \mathrm{i}(\rho_{s0} - \rho_{0s}) \qquad r_3 \equiv \rho_{ss} - \rho_{00}.$$

The atomic states are labelled $|0\rangle$ (ground) and $|s\rangle$ (excited); the energies of the states are E_0, E_s, and we define $\hbar\omega_s = E_s - E_0$ so that ω_s is the atomic resonance frequency. Then $\boldsymbol{\omega}(z, t) \equiv (\omega_1(z, t), 0, \omega_s)$ and $\omega_1(z, t) \equiv -2\, ex_{0s}\hbar^{-1}E(z, t)$; $ex_{0s} = ex_{s0}$ is the matrix element of the dipole operator ex. In equation (5) n is the number density of atoms and $n\, ex_{0s}r_1(z, t) \equiv nP(z, t)$ is then the dipole density. Equation (5) thus describes the propagation of strictly transverse fields (parallel to x): it neglects (I, II) Lorentz field contributions but this does not affect our conclusion on causality (i) (or on (ii) or (iii)).

The equations (4) and (5) follow from comparable operator equations only by decorrelating (see II and Bullough and Ahmad 1972a) but we believe our argument for causality extends to these (with qualifications about radiation damping). We can include correlation empirically by including damping and inhomogenous broadening: but these features do not really change our conclusions so we ignore them.

By reintroducing the magnetic field B we can put equation (5) into the quasilinear form

$$B_z - c^{-1}E_t - 4\pi nc^{-1}P_t = 0 \tag{6a}$$

$$E_z - c^{-1}B_t = 0 \tag{6b}$$

and equation (4) is (with $p \equiv ex_{0s}$)

$$(r_1)_t = -\omega_s r_2 \tag{7a}$$

$$(r_2)_t = \omega_s r_1 + 2pE\hbar^{-1}r_3 \tag{7b}$$

$$(r_3)_t = -2pE\hbar^{-1}r_2. \tag{7c}$$

Equations (6) and (7) constitute a system of five first order PDE like equation (1) in which all the coefficients $a^{\nu i}$ and $b^{\nu i}$ are constant. They are therefore semilinear.

The transformation matrix $t^{\mu\nu}$ can be derived by standard methods; but equations (7) already take the form of equation (2) (with $c^\nu = 0$) and it is simpler to combine (6) and (7*a*) with the definition of P to get (6) in characteristic form directly

$$(cE_z - E_t) + (cB_z - B_t) = 4\pi np\omega_s r_2 \tag{8a}$$

$$-(cE_z + E_t) + (cB_z + B_t) = 4\pi np\omega_s r_2. \tag{8b}$$

Equations (7) and (8) are now five equations in characteristic form. The characteristic lines, by equation (3), are simply $z = \pm ct$, $z = 0$; the last represents three coincident lines. The characteristic lines are fixed; and an important immediate deduction is that the system cannot create optical shocks. We look again at the problem of shock formation in § 4.

3. Causality

Since we have the characteristics, causality (i) follows without further analysis from a standard theorem due to Courant and Lax (1949). We start the system in the definite state $\{E(z, t_0), B(z, t_0), r_1(z, t_0), r_2(z, t_0), r_3(z, t_0)\}$ for some finite range of z (if there is a boundary the r_i are zero on the vacuum side and discontinuous across the boundary). We want to discover the state at a later space–time point (z', t'). Because we have chosen initial data on the line $t = t_0$ which is not a tangent at any point to the characteristics, the Courant–Lax theorem applies, and this in a manner best stated by reference to figure 1.

Select from the five one-parameter families of characteristics one from each passing through (z', t'). These cut the line $t = t_0$ at A, B, C in figure 1. The solution at (z', t') then depends, by the theorem, only on those initial data on $t = t_0$ between A and B. This proves causality (i). Next the extremum characteristics, $z = \pm ct$, form the boundary of the light cone; this proves (ii).

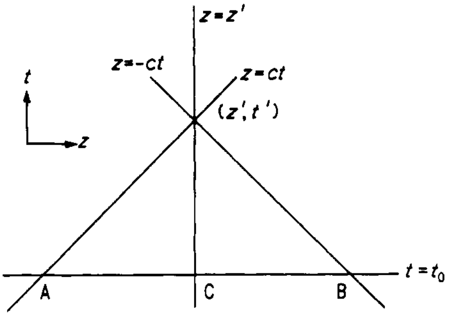

Figure 1. Graphical representation of the theorem of Courant and Lax (1949); the line $z = z'$ represents three degenerate characteristics.

We now prove (iii). By integrating equations (2) we obtain the result for the field $E(z, t)$ in terms of earlier fields

$$E(z,t) = 2\pi n\, e x_{0s}\omega_s c^{-1}\left\{\int_{z-ct}^{z} \mathrm{d}z' r_2\left(z', \frac{z'-z+ct}{c}\right) - \int_{z+ct}^{z} \mathrm{d}z' r_2\left(z', \frac{z-z'+ct}{c}\right)\right\} + \frac{1}{2}\{E(z-ct,0)+E(z+ct,0)-B(z-ct,0)+B(z+ct,0)\}. \quad (9)$$

If the pulse penetrates either a quiescent or a fully inverted dielectric, E, B and r_2 are zero before the pulse front (so defining that front). Then a front at $z-ct$ at $t = 0$ is at z at time t and travels with velocity c as stated in (iii). This result illustrates the point that initial discontinuities propagate along characteristics (Ames 1969).

Most of the current work in resonant nonlinear optics is in fact based on equations (4) and (5) and assumes approximate solutions consisting of slowly varying amplitudes modulating forward moving resonant carriers (McCall and Hahn 1969, Lamb 1971, I and II). This eliminates both 'back scattering' and the associated characteristic $z = -ct$. Initial data at times t in $t_1 > t \geqslant t_0$ which influence fields at (z_1, t) now lie in the right triangle defined by characteristics $z = z_1, z-ct = z_1-ct$, and the line $t = t_0$: the argument for causality still stands and shocks are excluded. This special case has already been noted by da Costa (1970).

4. Wavefronts and optical shocks

We have now completed the proofs of (i), (ii) and (iii) and the main argument of the paper. A number of comments by way of example can usefully be made now. Notice first that the velocity of the front is precisely c and that the characteristics $z = \pm ct$ are those of the vacuum even for propagation in a medium. This neither excludes distortionless nonlinear pulse propagation (as in II) at pulse velocities $V < c$ nor dispersion in linear theory. Linear theory actually follows from equations (4) and (5) by assuming fields so weak that $r_3(z,t) = r_3(z,t_0) =$ constant, as we next demonstrate. In this case we may then neglect equation (7c) and one characteristic but none of our conclusions are changed: in particular fronts propagate at c in the dispersive medium.

To examine linear theory we shall look for distortionless solutions $E(z \pm Vt)$, $r_1(z \pm Vt), r_2(z \pm Vt)$, of the equations (4) and (5) linearized by r_3 = constant = $r_3(t_0)$. We find that the only distortionless solutions are the trigonometrical functions $\cos \omega(z \pm Vt)$, $\sin \omega(z \pm Vt)$ which exist for each frequency ω. The amplitude E_0 of E is a free parameter. Necessary and sufficient conditions for these solutions are then

$$m^2(\omega) - 1 = -\frac{8\pi n\, e^2 x_{0s}^2 \omega_s r_3(t_0)}{\hbar(\omega_s^2 - \omega^2)}$$

$$R_0 = \frac{m^2(\omega) - 1}{4\pi n\, e x_{0s}} E_0 \tag{10}$$

where

$$m = cV^{-1} \quad \text{and} \quad r_1(z \pm Vt) = R_0 \begin{cases} \cos \omega(z \pm Vt) \\ \sin \omega(z \pm Vt) \end{cases}.$$

These distortionless solutions propagate along lines $z = \pm Vt$. In the 'attenuator' the dielectric is prepared everywhere with $r_3(t_0) < 0$: if the atoms are all actually in their ground states initially $r_3(t_0) = -1$ everywhere and the dielectric is totally quiescent. Thus in the attenuator if $\omega < \omega_s$, $m^2(\omega) > 1$ and $V < c$, the lines $z = \pm Vt$ lie within the light cone. If however either $r_3(t_0) > 0$, as in the 'amplifier', where for example $r_3(t_0) = +1$ describes a dielectric with all atoms initially inverted, and $\omega < \omega_s$, or $r_3(t_0) < 0$ and $\omega > \omega_s$, $m^2(\omega) < 1$, $V > c$ and the lines $z = \pm Vt$ are outside the light cone.

In practice $E(z \pm Vt)$ penetrates the dielectric from outside with a front on $z \pm ct$ = constant. Our result (iii) then shows that this propagates with velocity c inside the dielectric so that the front of $E(z \pm Vt)$ actually distorts. Apparently it steepens if it can in an amplifier below resonance since $V > c$ and waves tend to pile up; the front flattens in an attenuator since the front outstrips the wave train. The opposite is the case for a tail: in the attenuator below resonance, for example, the tail catches up with the wave train. Because the characteristics are fixed no discontinuities, that is shocks, can build up however.

These results in a dispersive linear medium are the familiar result that group and wave velocities are not synonymous. But an explanation in these terms becomes inadequate in nonlinear theory where the trigonometric functions are replaced (cf II) by Jacobian elliptic functions. Waves are now wave groups and the distortionless condition means that every harmonic wave within the group has the same velocity V, and the group velocity is now V. In practice these wave groups, which have 'infinite support' (in the mathematical sense of this ie do not vanish everywhere outside some finite region of z at fixed t) could only be approximated experimentally by wave groups of finite support; they have fronts and tails containing discontinuities and these will travel at velocity c. One tentative conclusion so far from our numerical work on pulses in the amplifier however is that average pulse profiles flatten rather than steepen; the front can appear as a rapidly changing extra peak at the front of the profile. This is probably a nonlinear effect.

Away from resonance the linear medium is dispersionless and m^2 is constant. The equations (4) and (5) reduce to the pair of first order PDE equivalent to the second order wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 (m^2 E)}{\partial t^2} = 0. \tag{11}$$

The characteristics are now $z \pm cm^{-1}t$ = constant and these two approximate sets of characteristics replace the four characteristic families $z \pm ct$ = constant, $z = 0$.

In the case of nonlinear distortionless pulse propagation there is (cf II) a 'refractive index' for each Jacobian elliptic function labelled by its frequency ω, and this refractive index depends on pulse power. A rough argument now is to say that an equation like (11) is applicable in the general nonresonant, nonlinear case with m^2 a function of $E^2(z, t)$. This system of equations is now nonlinear but no longer semilinear: the characteristics are movable and this means (cf eg Forsythe and Wasow 1960) that shock formation is possible.

To see this explicitly note that equation (11) can be set in the form

$$cB_z + f(E)E_t = 0 \tag{12a}$$

$$cE_z + B_t = 0 \tag{12b}$$

with $f(E) \equiv \mathrm{d}(Em^2(E))/\mathrm{d}E$. These equations can immediately be set in the characteristic form (assuming $f > 0$)

$$(f^{1/2}(E)E_t \pm cE_z) \pm f^{-1/2}(E)(f^{1/2}(E)B_t \pm cB_z) = 0 \tag{13a, b}$$

and characteristic lines are

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \pm cf^{-1/2}(E) \tag{14}$$

plainly depending on the solution E. Integrals along the characteristics themselves are

$$G(E(z, t)) \pm B(z, t) = \text{constant} \tag{15a}$$

where

$$G(E(z, t)) \equiv \int^{E(z,t)} f^{1/2}(E')\,\mathrm{d}E'. \tag{15b}$$

Thus in general

$$G(E(z, t)) = \tfrac{1}{2}\{G(E(z_1, t_0)) + G(E(z_2, t_0)) + B(z_1, t_0) - B(z_2, t_0)\} \tag{16}$$

where (z_1, t_0) and (z_2, t_0) are points on the intersection of the line $t = t_0$ with the two characteristics from the different families of (14) passing through (z, t). If, for example, we ignore back scattering, that is we ignore one family of characteristics, $G(E(z, t))$ is constant along the lines of the second family and $E(z, t)$ is constant also. The family is therefore the family of straight lines

$$(z - z_0) - cf^{-1/2}(E(z_0, t_0))(t - t_0) = 0. \tag{17}$$

The members of this family may intersect.

Intersection between lines through (z_0, t_0) and (z_0', t_0) occurs when

$$\begin{aligned} z &= z_0 + cf^{-1/2}(E(z_0, t_0))(t - t_0) \\ &= z_0' + cf^{-1/2}(E(z_0', t_0))(t - t_0). \end{aligned} \tag{18}$$

This intersection will happen after the time $(t-t_0)$, therefore, if

$$(t-t_0)^{-1} = \frac{c\{f^{-1/2}(E(z_0, t_0))-f^{-1/2}(E(z_0', t_0))\}}{z_0-z_0'}. \tag{19}$$

Thus *adjacent* lines cross after a time $(t-t_0)$ given by

$$(t-t_0)^{-1} = c\frac{\mathrm{d}}{\mathrm{d}z_0}f^{-1/2}(E(z_0, t_0)) = -\frac{1}{2}cf^{-3/2}(E(z_0, t_0))f''(E)\frac{\mathrm{d}E(z_0, t_0)}{\mathrm{d}z_0}. \tag{20}$$

This actually occurs instantaneously, that is on the line $t = t_0$, when $\mathrm{d}E/\mathrm{d}z_0$ is undefined (ie it is essentially infinite). The profile for fixed t is now singular but perhaps no more than discontinuous. This is usually taken as the criterion for shock formation.

It is noteworthy that the results (14) to (20) include those obtained by DeMartini *et al* (1967). Suppose $m^2(E) = m_0^2+\lambda E^2$ so that $f(E) = m_0^2+3\lambda E^2$. Characteristic lines without back scattering are

$$(z-z_0) = \frac{c(t-t_0)}{(m_0^2+3\lambda E^2(z_0, t_0))^{1/2}} \sim \frac{c}{m_0}\left(1-\frac{3}{2m_0^2}E^2(z_0, t_0)\right)(t-t_0). \tag{21}$$

These are the characteristics derived by DeMartini *et al* (1967) from consideration of energy flow.

There is no actual paradox in the result that equations (12) yield optical shocks whilst equations (4) and (5) do not. Equations (12) do *not* follow from equations (4) and (5) since these equations show that m^2, that is $P(z, t)$, is *not* a function of $E(z, t)$. Instead $P(z, t)$ is a nonlinear *functional* of $E(z, t)$. It seems to be this, rather than the neglect of back scattering, which leads to the two very different conclusions. The conclusion that no shocks are possible is based on the more fundamental set of equations and would appear to be the proper physical conclusion. However, it is an open question whether it is nevertheless good physics to assume equations (12) rather than equations (4) and (5) for a description of optical pulse propagation on a scale of say 10^{-12} s or longer. The natural time scale of equations (4) and (5) is a reciprocal optical cycle ($\sim 10^{-15}$ s). A shock on a 'macroscopic' scale of 10^{-12} s is not necessarily a true shock on a 10^{-15} s time scale, nor may the pulse even be particularly steep.

It remains true that shock formation on a microscopic scale is still possible if there are additional sources: Čerenkov radiation for example can be induced from equations (4) and (5) even linearly by incident electrons. Further, once radiation reaction is included, the equations (4) and (5) are no longer semilinear and true shock formation on a microscopic scale becomes possible†. It should be added also that because $P(z, t)$ is a functional of $E(z, t)$, m^2 cannot be dispersionless. DeMartini *et al* (1967) note that dispersion smooths out their shocks. Dispersion also complicates and apparently destroys the linear optical shock accompanying Čerenkov radiation (J D Gibbon 1971, private communication).

† Within semiclassical theory one particle radiation is nonlinear as discussed by Stroud and Jaynes (1970) and Crisp and Jaynes (1969). The solution of nonlinear *second quantized* theory is identical with that of linear theory, however (Bullough and Ahmad 1972b). Nonlinear damping in a dielectric of many coupled atoms needs separate discussion: Dicke (1954) described 'super-radiant' enhancement of the radiation rate by coherent coupling of the spontaneous emission from many atoms and envisaged (Dicke 1964) an 'optical bomb' exhibited as some form of optical shock. On the other hand inclusion of semiclassical nonlinear damping for an inverted (amplifying) many atom dielectric offers the possibility of a different solution, namely of a distortionless self-sustained (ie undamped) oscillatory solution (Bullough and Ahmad 1972b).

5. Initial conditions

We now look at the problem of initial data. Nonlinear theories based on equations (4) and (5) are unusual amongst physical theories in that the question of a correct choice of initial data is an important one. Most current theoretical conclusions on short pulses are based on approximations to equations (4) and (5) applied to nonphysical infinite dielectrics (eg Icsevgi and Lamb 1969, Lamb 1971, Hopf and Scully 1969). Since all physical dielectrics are finite with surfaces, any self-consistent system of fields and dipoles inside an infinite dielectric may be impossible to introduce into a finite dielectric. The problem of pulse speeds $V > c$ in an amplifier (eg that of the hyperbolic secant pulse modulating a carrier (McCall and Hahn 1969)) is typical of infinite pulses and infinite dielectrics. Icsevgi and Lamb (1969) and da Costa (1970) correctly analyse this case and our result (iii) shows within either linear or nonlinear theory that if the pulse has finite support (that is a physical or at least a numerically calculable pulse) the front travels at precisely velocity c. But we must add that it is dangerous to provide initial data partly or wholly on characteristics (as eg Icsevgi and Lamb 1970, Burnham and Chiao 1969): a condition for the Courant–Lax theorem is broken, the proof of causality (i) does not follow and even uniqueness of the solution becomes suspect.

In the case where initial data lie along one or more characteristics, uniqueness can be proved only in certain special cases: in some cases uniqueness can be disproved. Ames (1969) gives a discussion for two quasilinear PDE with two characteristics. The solution is not unique for initial data given only on one characteristic (cf Icsevgi and Lamb 1969 § VA). Uniqueness can be restored by giving data consistently on both characteristics. For more than two equations the only theorem we know is due to Hörmander (1964) and states that as above initial data given only on one characteristic are insufficient for uniqueness.

6. Some numerical results

The problems of initial data discussed in § 5 do not arise for initial data given on $t = t_0$ and for physical (ie finite) dielectrics. We have integrated (7) and (8) numerically, with pulses entering dielectrics confined to $z > 0$ or $0 < z < a$ (slab), using finite difference methods on the grid of characteristic lines (cf Ames 1969). It seems worthwhile quoting here some preliminary numerical results.

6.1. Back scattering

An amplifying dense dielectric can back scatter a fairly large fraction of the incident field amplitude. This result is important, for back scattering is certainly ignored in previous work on resonant media (cf Lamb 1971). Of course a linear medium may back scatter up to 100% of the incident field: our nonlinear example is 15% back scattering of amplitude for 10^{22} atoms $\mathrm{cm^{-3}}$. The pulse is exceptionally short ($\sim 10^{-14}$ s) and very intense (10^{11} $\mathrm{W\,cm^{-2}\,s^{-1}}$), but since pulse velocities depend on the power of the pulse we can expect a reduction in back scattering as compared to the linear medium.

6.2. DC field

The nonlinear optical equations have analytical steady-state DC solutions. When an external DC field is applied to a dielectric there will be a transition period during which the dielectric evolves from the quiescent state to the equilibrium DC state. We find that during this transition period the dielectric emits light at resonant frequency†.

As a numerical example, a DC field of 3×10^6 V cm^{-1} applied to a slab of dielectric of atomic density 10^{21} cm^{-3} and thickness 3×10^{-3} cm produced an output wave of $E_0\{1+\sin \omega_s(t-z/c)\}$ with E_0 constant over about 6×10^{-3} cm and equal to about 2·4 % of the applied DC field. The result needs further study.

6.3. Sech pulses

Although the hyperbolic secant pulse solution of I and II propagates without distortion (as predicted) in an attenuator, the same pulse with any small imperfections ($\sim$0·1 %) breaks up rapidly in an amplifier; the pulse front travels at c precisely and no physical meaning can be attached to pulse speeds greater than c in agreement with (ii) and (iii) above (from I and II we know that a *perfect* hyperbolic secant which has infinite support, will travel in an amplifier without distortion at a speed $V > c$). Again the pulse is very short ($\sim 10^{-14}$ s).

6.4. Arbitrary pulses

Pulses which are not travelling wave solutions of the equations (7) and (8) break up into approximately sinusoidal forms, with no sign of evolution to a stable form within the short time scale in which we have observed them ($\sim 10^{-13}$ s). At low densities (10^{18} cm^{-3}) the pulse is attenuated slowly: each atom 'sees' the same pulse (cf Burnham and Chiao 1969) and the pulse leaves a trail of constant $r_3(z, t)$ over a relatively large distance. (Recall that r_3 is a measure of the atomic inversion.) At higher densities (10^{22} cm^{-3}) the pulse is attenuated rapidly and this effect does not occur. The density $n \sim 10^{18}$ cm^{-3} also heralds the breakdown of the slowly varying amplitude approximation in the theory of self-induced transparency (Bullough and Ahmad 1972a).

6.5. Colliding pulses

The negative but perhaps surprising result that two ultrashort optical pulses traversing a dielectric slab in opposite directions may do so without significant interaction during overlap at particle densities as high as 10^{22} cm^{-3}. (This may be a result of the very short collision time.) There appears to be a direct analogy between these (sech) pulses and the 'soliton' pulses discovered by Kruskal and Zabusky (Zabusky 1967) in their work on the anharmonic lattice and the Korteweg–de Vries equation.

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† We have not proved that the system actually evolves to a steady time independent state since we have not included radiation reaction.

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